MATH4210: Financial Mathematics

II. Probability theory review and Brownian motion

Probability space

A triple $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a probability space, if

- Ω is a (sample) sapce,
- $\mathcal{F}$ is a σ-field on Ω,
  - $\Omega \in \mathcal{F}$,
  - $A \in \mathcal{F} \implies A^c \in \mathcal{F}$,
  - $A_i \in \mathcal{F}, \ \forall i = 1, 2, \cdots \implies \cup_{i=1}^{\infty} A_i \in \mathcal{F}$.
- $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$,
  - $\mathbb{P}[A] \in [0, 1]$ for all $A \in \mathcal{F}$.
  - $\mathbb{P}[\Omega] = 1$.
  - Let $A_i \in \mathcal{F}, \ \forall i = 1, 2, \cdots$ and such that $A_i \cap A_j = \emptyset$ if $i \neq j$, then $\mathbb{P}\big[\cup_{i=1}^{\infty} A_i \in \mathcal{F}\big] = \sum_{i=1}^{\infty} \mathbb{P}[A_i]$.

Random variable, expectation

- A random variable is a map $X : \Omega \to \mathbb{R}$ such that

$$\{X \leq c\} := \{\omega \in \Omega \ : X(\omega) \leq c\} \in \mathcal{F}, \quad \forall c \in \mathbb{R}.$$

- Law of X:
  - Distribution function

$$F(x) := \mathbb{P}[X \leq x].$$

  - Density function (if X follows a continuous law)

$$\rho(x) = F'(x), \qquad \rho(x)dx = \mathbb{P}\big[X \in [x, x + dx]\big].$$

  - Characteristic function

$$\Phi(\theta) := \mathbb{E}\big[e^{i\theta X}\big]$$

Random variable, expectation

- Expectation:

$$\mathbb{E}\big[f(X)\big] = \int_{\mathbb{R}} f(x)\rho(x)dx.$$

- Variance:

$$\mathrm{Var}\big[f(X)\big] = \mathbb{E}\big[f(X)^2\big] - \big(\mathbb{E}[f(X)]\big)^2.$$

- Co-variance:

$$\mathrm{Cov}\big[X, Y\big] = \mathbb{E}\big[XY\big] - \mathbb{E}[X]\mathbb{E}[Y].$$

- $X_1, \cdots, X_n$ are mutually independent if

$$\mathbb{E}\Big[\prod_{i=1}^{n} f_k(X_k)\Big] = \prod_{k=1}^{n} \mathbb{E}\big[f_k(X_k)\big], \quad \forall \text{ bounded measurable functions } f_k.$$

Remark: If $X_1, \cdots, X_n$ are mutually independent, then $f_1(X_1), \cdots, f_n(X_n)$ are also mutually independent.

Conditional expectation

- Let $A, B \in \mathcal{F}$ such that $\mathbb{P}[B] > 0$, we define the conditional probability of A knowing B by

$$\mathbb{P}\big[A\big|B\big] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}.$$

- Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$ (i.e. $\mathcal{G} \subset \mathcal{F}$ and $\mathcal{G}$ is a σ-field), and X be a random variable such that $\mathbb{E}[|X|] < \infty$, we define the conditional expectation of X knowing $\mathcal{G}$ as the random X such that

$$\mathbb{E}[|Z|] < \infty, \qquad Z \text{ is } \mathcal{G}\text{-measurable},$$

and

$$\mathbb{E}[YX] = \mathbb{E}[YZ], \ \forall\ \mathcal{G}\text{-measurable bounded random variables } Y.$$

Denote the conditional expectation: $\mathbb{E}[X|\mathcal{G}] := Z$.

- Denote the condition probability $\mathbb{P}[A|\mathcal{G}] := \mathbb{E}[1_A|\mathcal{G}]$.

Conditional expectation

• A random variable Y is $\mathcal{G}$-measurable if $\{Y \leq c\} \in \mathcal{G}$ for all $c \in \mathbb{R}$. Further, Y is $\mathcal{G}$-measurable implies that $f(Y)$ is $\mathcal{G}$-measurable for all bounded measurable functions f.

• Let Y be a random variable, we denote by $\sigma(Y)$ the smallest σ-field $\mathcal{G}$ such that Y is $\mathcal{G}$-measurable Denote then

$$\mathbb{E}[X|\sigma(Y)] = \mathbb{E}[X|Y].$$

Conditional expectation

Remember the following proposition !

Proposition 1.1

(i). *Tower property:* $\mathbb{E}\big[\mathbb{E}[X|\mathcal{G}]\big] = \mathbb{E}[X]$.
(ii). $\mathbb{E}[XY|\mathcal{G}] = X\mathbb{E}[Y|\mathcal{G}]$ *if* X *is* $\mathcal{G}$*-measurable.*
(iii). $\mathbb{E}[X|Y] = \mathbb{E}[X]$ *if* X *is independent of* Y.
(iv). $\mathbb{E}\big[\alpha X + \beta Y \big| \mathcal{G}\big] = \alpha\mathbb{E}[X|\mathcal{G}] + \beta\mathbb{E}[Y|\mathcal{G}]$.

Brownian Motion

Brownian motion is a stochastic process $B = (B_t)_{t\geq 0}$, where each B_t is a random variable.

• Historically speaking, Brownian motion was observed by Robert Brown, an English botanist, in the summer of 1827, that "pollen grains suspended in water performed a continual swarming motion." Hence it was named after Robert Brown, called Brownian motion.

• In 1905, Albert Einstein gave a satisfactory explanation and asserted that the Brownian motion originates in the continued bombardment of the pollen grains by the molecules of the surrounding water, with successive molecular impacts coming from different directions and contributing different impulses to the particles.

• In 1923, Norbert Wiener (1894-1964) laid a rigorous mathematical foundation and gave a proof of its existence. Hence, it explains why it is now also called a Wiener process. In the sequel, we will use both Brownian motion and Wiener process interchangeably.

Brownian Motion

We say a stochastic process $(B_t)_{t\geq 0}$ is a standard *Brownian motion* or *Wiener process* on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ if it satisfies the following conditions:

(a) $B_0 = 0$.

(b) the map $t \mapsto B_t$ is continuous for $t \geqslant 0$.

(c) stationary increment: the change $B_t - B_s$ is normally distributed: $N(0, t-s)$ for all $t > s \geqslant 0$.

(d) independent increment: the changes $B_{t_1} - B_{t_0}$, $B_{t_2} - B_{t_1}, \cdots,$ $B_{t_{n+1}} - B_{t_n}$ are mutually independent for all $0 \leqslant t_0 < t_1 < \cdots < t_n < t_{n+1}$.

- We accept that the Brownian motion exists.

Brownian Motion

Example 2.1

Given a set $A \subseteq \mathbb{R}$, compute the probability $\mathbb{P}(B_{t+h} - B_t \in A)$, $h > 0$.

Because $B_{t+h} - B_t \sim N(0, h)$, we have

$$\mathbb{P}(B_{t+h} - B_t \in A) = \frac{1}{\sqrt{2\pi h}} \int_A e^{-\frac{x^2}{2h}} \, \mathrm{d}x.$$

Brownian Motion

Markov property

Which means that only the present value of a process is relevant for predicting the future, while the past history of the process and the way that the present has emerged from the past are irrelevant.

$$\mathbb{P}(X_u \in A \mid X_s, 0 \leqslant s \leqslant t) = \mathbb{P}(X_u \in A \mid X_t), \quad \forall\, u \geqslant t \geqslant 0.$$

Markov process

A stochastic process which satisfies the Markov property is called a Markov process.

Brownian motion is a Markov process because by property (d) $B_u - B_t$ is independent of $B_s - B_0 = B_s$ for all $0 \leqslant s \leqslant t \leqslant u$.

Brownian Motion

Example 2.2

Given a set $A \subseteq \mathbb{R}$ and all the information up to time t, compute the conditional probability $\mathbb{P}(B_{t+h} \in A \mid B_s, 0 \leqslant s \leqslant t)$, $h > 0$.

Brownian motion is a Markov process, so we have

$$\begin{aligned} & \mathbb{P}(B_{t+h} \in A \mid B_s, 0 \leqslant s \leqslant t) \\ = & \ \mathbb{P}(B_{t+h} \in A \mid B_t) \\ = & \ g(B_t), \end{aligned}$$

where

$$g(x) = \mathbb{P}[B_{t+h} - B_t \in A_x \mid B_t = x], \qquad A_x := \{y - x \ : y \in A\}.$$

Brownian Motion

Filtration

A filtration $(\mathcal{F}_t)_{t\geq 0}$ is a family of σ-field such that $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$, for all $s \leq t$.

Martingale

A process $(X_t)_{t\geqslant 0}$ is called a martingale with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$, if $\mathbb{E}[|X_t|] < \infty$ for all $t \geq 0$, and

$$\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s, \qquad \forall\, 0 \leqslant s \leqslant t.$$

Brownian Motion

Let us set $\mathcal{F}_t := \sigma\{B_s, s \leq t\}$.

Proposition 2.1

We have

- $\mathrm{Cov}(B_s, B_t) = \min(s, t)$.
- $\{B_t, t \geqslant 0\}$ *is a martingale.*
- $\{B_t^2 - t, t \geqslant 0\}$ *is a martingale.*
- $\{e^{aB_t - a^2 t/2}, t \geqslant 0\}$ *is a martingale.*

Quadratic variation of Brownian motion

• Let $\Delta_n = (0 = t_0^n < t_1^n < \cdots < t_n^n = t)$ a subdivision of interval $[0, t]$, where $t_k^n := k\Delta t$ with $\Delta t = t/n$. Let

$$Z_t^n := \sum_{k=0}^{n-1} \left(B_{t_k^n} - B_{t_{k-1}^n}\right)^2 \text{ and } V_t^n := \sum_{k=0}^{n-1} \left|B_{t_k^n} - B_{t_{k-1}^n}\right|.$$

Proposition 2.2

Let $n \longrightarrow \infty$, then

$$Z_t^n \longrightarrow t, \quad \text{and} \quad V_t^n \longrightarrow \infty, \quad \text{in probability.}$$

The heat equation

- Define

$$q(t,x,y) = \frac{1}{\sqrt{2\pi t}} \exp\Big(- \frac{(y-x)^2}{2t} \Big).$$

Proposition 2.3

The function $q(t,x,y)$ *satisfies*

$$\partial_t q(t,x,y) = \frac{1}{2}\partial^2_{xx} q(t,x,y) = \frac{1}{2}\partial^2_{yy} q(t,x,y).$$

The heat equation

Theorem 2.1

Let $f : \mathbb{R} \to \mathbb{R}$ *be such that, for some constant* $C > 0$, $f(x) \le e^{C|x|}$ *for all* $x \in \mathbb{R}$. *Define*

$$u(t,x) \;:=\; \mathbb{E}\big[f(B_T)\big|B_t = x\big] \;=\; \mathbb{E}\big[f(B_T - B_t + x)\big].$$

Then u *satisfies the heat equation*

$$\partial_t u + \frac{1}{2}\partial^2_{xx} u = 0,$$

with terminal condition $u(T,x) = f(x)$.

Generalized Brownian Motion

Give real scalars x_0, μ, and $\sigma > 0$, we call process

$$X_t = x_0 + \mu t + \sigma B_t$$

as *generalized Brownian motion* or *Brownian motion with drift* or *generalized Wiener process* starting at x_0, with a *drift* rate μ and a *variance* rate σ^2.

Let us denote

$$\begin{cases} \mathrm{d}X_t = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B_t, \\ X_0 = x_0. \end{cases}$$

By the property (c) of Brownian motion, we have

$$X_{t+h} - X_t \sim N(\mu h, \sigma^2 h).$$

Generalized Brownian motion and PDE

Theorem 2.2

Let $f : \mathbb{R} \to \mathbb{R}$ *be such that, for some constant* $C > 0$, $f(x) \leq e^{C|x|}$ *for all* $x \in \mathbb{R}$. *Define*

$$u(t,x) \;:=\; \mathbb{E}\big[f(X_T)\big|X_t = x\big] \;=\; \mathbb{E}\big[f(X_T - X_t + x)\big].$$

Then u *satisfies the heat equation*

$$\partial_t u + \mu \partial_x u + \frac{1}{2}\sigma^2 \partial^2_{xx} u = 0,$$

with terminal condition $u(T,x) = f(x)$.

Stochastic Integral: simple process

Let $\theta : [0,T] \times \Omega \to \mathbb{R}$ be a simple process, i.e.

$$\theta_t = \alpha_0 \mathbf{1}_{\{0\}}(t) + \sum_{i=0}^{n-1} \alpha_i \mathbf{1}_{(t_i, t_{i+1}]}(t) = \begin{cases} \alpha_0, & t = 0; \\ \alpha_k, & t_k < t \leqslant t_{k+1}, \end{cases}$$

for a discret time grid $0 = t_0 < t_1 < \cdots < t_n = T$ and bounded $\mathcal{F}_{t_i}$-measurable random variables α_i, we define

$$\begin{aligned} \int_0^T \theta_t \, \mathrm{d}B_t =& \sum_{i=0}^{n-1} \alpha_i (B_{t_{i+1}} - B_{t_i}) \\ =& \alpha_0 (B_{t_1} - 0) + \alpha_1 (B_{t_2} - B_{t_1}) + \cdots + \alpha_{n-1} (B_{t_n} - B_{t_{n-1}}). \end{aligned}$$

Stochastic Integral: simple process

Theorem 3.1 (Itô Isometry)

Let θ be a simple process, then

$$\mathbb{E}\Big[\int_0^T \theta_t \,\mathrm{d}B_t\Big] = 0, \quad \textit{and} \quad \mathbb{E}\Big[\Big(\int_0^T \theta_t \,\mathrm{d}B_t\Big)^2\Big] = \mathbb{E}\Big[\int_0^T \theta_t^2 \,\mathrm{d}t\Big].$$

Stochastic Integral

We accept the following facts in mathematics:

Let $\mathbb{L}^2(\Omega)$ denote the space of all square integrable random variables $\xi : \Omega \to \mathbb{R}$, $\mathbb{H}^2[0,T]$ be the set of $(\mathcal{F}_t)_{t\geq 0}$-adapted right-continuous and left-limit processes θ, such that

$$\mathbb{E}\Big[\int_0^T \theta_t^2 \,\mathrm{d}t\Big] < +\infty.$$

- $\mathbb{L}^2(\Omega)$ and $\mathbb{H}^2[0,T]$ are both Hilbert space.
- In a Hilbert space E, let $(e_n)_{n\geq 1}$ be a Cauchy sequence, i.e. $|e_m - e_n| \to 0$ as $m, n \to \infty$, then there exists a unique $e_\infty \in E$ such that $e_n \to e_\infty$ as $n \to \infty$.
- For each $\theta \in \mathbb{H}^2([0,T])$, there exists a sequence of simple processes $(\theta^n)_{n\geq 1}$ such that

$$\lim_{n\to\infty} \mathbb{E}\Big[\int_0^T \left(\theta_t - \theta_t^n\right)^2 \mathrm{d}t\Big] = 0.$$

Stochastic Integral

Let $\theta \in \mathbb{H}^2([0,T])$, $(\theta^n)_{n\geq 1}$ be a sequence of simple processes such that

$$\lim_{n\to\infty} \mathbb{E}\Big[\int_0^T \big(\theta_t - \theta_t^n\big)^2 \,\mathrm{d}t\Big] = 0.$$

Proposition 3.1

The sequence of stochastic integrals $\int_0^T \theta_t^n \,\mathrm{d}B_t$ *has a unique limit in* $\mathbb{L}^2(\Omega)$ *as* $n \to \infty$.

- Let us define

$$\int_0^T \theta_t \,\mathrm{d}B_t \;:=\; \lim_{n\to\infty} \int_0^T \theta_t^n \,\mathrm{d}B_t.$$

Stochastic Integral

Theorem 3.2 (Itô Isometry)

Let $\theta \in \mathbb{H}^2[0,T]$, *then*

$$\mathbb{E}\Big[\int_0^T \theta_t \, \mathrm{d}B_t\Big] = 0, \quad \textit{and} \quad \mathbb{E}\Big[\Big(\int_0^T \theta_t \, \mathrm{d}B_t\Big)^2\Big] = \mathbb{E}\Big[\int_0^T \theta_t^2 \, \mathrm{d}t\Big].$$

Itô's Lemma

Theorem 3.3 (Itô's Lemma)

Let B be a Brownian motion, $f(t,x)$ a smooth function. Then the process $Y_t = f(t, B_t)$ is also an Itô process and

$$Y_t \;=\; Y_0 + \int_0^t \left(\partial_t f(s,B_s) + \tfrac{1}{2}\partial^2_{xx} f(s,B_s)\right) \mathrm{d}s \;+\; \int_0^t \partial_x f(s,B_s)\,\mathrm{d}B_s.$$

or equivalently,

$$\mathrm{d}Y_t \;=\; \left(\partial_t f(t,B_t) + \tfrac{1}{2}\partial^2_{xx} f(t,B_t)\right) \mathrm{d}t \;+\; \partial_x f(t,B_t)\,\mathrm{d}B_t.$$

A technical Lemma

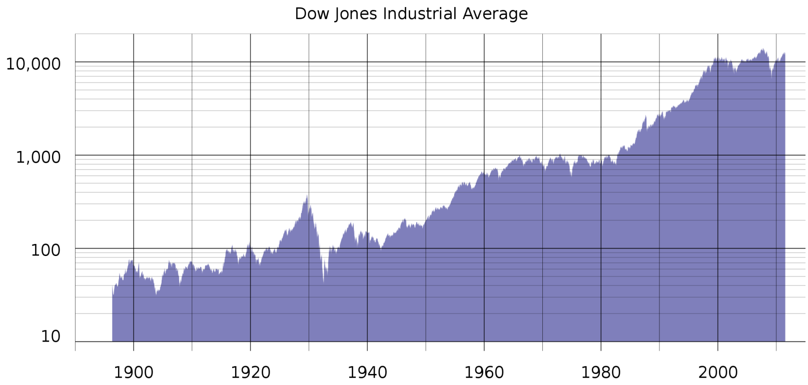
• Let $\Delta_n = (0 = t_0^n < t_1^n < \cdots < t_n^n = t)$ a subdivision of interval $[0, t]$, where $t_k^n := k\Delta t$ with $\Delta t = t/n$.

Lemma 3.1

Let H be a (uniformly bounded) adapted process (i.e. $H_s \in \mathcal{F}_s$ for all $s \geq 0$), denote $\Delta B_{k+1}^n := B_{t_{k+1}^n} - B_{t_k^n}$. Then

$$\sum_{k=0}^{n-1} H_{t_k^n}\left((\Delta B_{k+1}^n)^2 - \Delta t\right) \longrightarrow 0, \text{ in probability as } n \longrightarrow \infty.$$

Samples of Stock Price



Samples of Stock Price

Stock price of Hong Kong Electric from 2006 to 2011

scource: http://hk.finance.yahoo.com/

Model of Stock Price

Key observation

- Stock price has a trend. drift
- We see fluctuation of the stock price. volatility

Model of Stock Price

We will build our model of the stock price based on the above observation. We first recall the (relative) return is defined to be the change in the price divided by the original value,

$$\text{relative return} = \frac{\text{price} - \text{original price}}{\text{original price}}.$$

The (relative) return of stock on a short time $[t, t + \Delta t]$ is expressed as

$$\frac{\Delta S_t}{S_t},$$

where $\Delta S_t = S_{t+\Delta t} - S_t$. By the above observation,

$$\frac{\Delta S_t}{S_t} = \mu \Delta t + \sigma \Delta B_t,$$

where μ is the drift of the stock, σ is the volatility of the stock, ΔB_t is a random variable with zero mean.

Model of Stock Price

Let us adopt the differential notation used in calculus. Namely, we use the notation $\mathrm{d}t$ for the small change in any quantity over this time interval when we intend to consider it as an infinitesimal change. We obtain a *stochastic differential equation (SDE)*

$$\frac{\mathrm{d}S_t}{S_t} = \mu \,\mathrm{d}t + \sigma \,\mathrm{d}B_t.$$

It can also be expressed as

$$\mathrm{d}S_t = \mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}B_t,$$

or

$$S_t = S_{t_0} + \int_{t_0}^{t} \mu S_u \,\mathrm{d}u + \int_{t_0}^{t} \sigma S_u \,\mathrm{d}B_u.$$

Continuous-Time Model of Stock Price

- Let $f(t,x) = Ce^{bt+cx}$ for some constant C, b, c, then

$$\partial_t f(t,x) = bf(t,x), \quad \partial_x f(t,x) = cf(t,x), \quad \partial_{xx}^2 f(t,x) = c^2 f(t,x).$$

It follows from Itô's Lemma that $S_t = f(t, B_t)$ satisfies

$$\begin{aligned} \mathrm{d}S_t = \mathrm{d}f(t,B_t) &= \Big(\partial_t(t,B_t) + \frac{1}{2}\partial_{xx}^2 f(t,B_t)\Big)\,\mathrm{d}t \\ &\quad + \partial_x f(t,B_t)\,\mathrm{d}B_t \\ &= \big(b + \tfrac{1}{2}c^2\big) S_t\,\mathrm{d}t + cS_t\,\mathrm{d}B_t. \end{aligned}$$

Continuous-Time Model of Stock Price

- The Black-Scholes model:

$$\mathrm{d}S_t = \mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}B_t.$$

We obtain that

$$C = S_0, \qquad b = \mu - \frac{1}{2}\sigma^2, \qquad c = \sigma,$$

so that

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t},$$

we call the process $S(\cdot)$ as a *geometric Brownian motion (GBM)*.

Continuous-Time Model of Stock Price

From

$$\ln(S_t) = \ln(S_0) + \left(\mu - \tfrac{1}{2}\sigma^2\right) t + \sigma B_t,$$

we see that

$$\ln(S_T) - \ln(S_0) \sim N\left(\left(\mu - \tfrac{1}{2}\sigma^2\right) T, \sigma^2 T\right),$$

or

$$\ln(S_T) \sim N\left(\ln(S_0) + \left(\mu - \tfrac{1}{2}\sigma^2\right) T, \sigma^2 T\right).$$

We say S_T follows a *log-normal distribution* because the $\log$ of S_T is normally distributed.